THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS
MATH2050 (First Term
Mathematical Analysis I
Homework VI

Questions with * will be marked.

1. Let $f : A \to \mathbb{R}$, $A^+ = A \cap (x_0, \infty)$, $A^- = A \cap (-\infty, x_0)$, where $x_0 \in \mathbb{R}$ is clustered (=non-isolated) with respect to A^+ and A^-.
   Show that $\lim_{x \to x_0} f(x) = \ell (\in \mathbb{R})$ if and only if $\lim_{x \to x_0^+} f(x) = \ell = \lim_{x \to x_0^-} f(x)$.
   *State and prove*the corresponding result for $\ell = +\infty$ (and also that for $\ell = -\infty$).

2. Let $g : (a, \infty) \to (b, \infty)$ and $f : (b, \infty) \to \mathbb{R}$. Suppose that $\lim_{x \to \infty} f(x) = \ell$ and $\lim_{t \to \infty} g(t) = \infty$. Show that $\lim_{t \to \infty} f(g(t)) = \ell$.

3. Let $g : (a, b) \to (c, d)$ and $f : (c, d) \to \mathbb{R}$, and let $t_0 \in (a, b)$, $x_0 \in (c, d)$ be such that $\lim_{t \to t_0} g(t) = x_0$. Suppose that $\lim_{x \to x_0} f(x) = \ell (\in \mathbb{R})$. In view of question 2 above, it is natural to ask: true or not that $\lim_{t \to t_0} f(g(t)) = \ell$?

   (a) If there exists $\gamma > 0$ such that $g(t) \neq x_0$ for all $t \in V_\gamma(t_0) \setminus \{t_0\}$, then prove "yes".

   (b) Give a counter example, otherwise.

4. Evaluate the limits (if exist) or show that they do not exist.

   (a) $\lim_{x \to 1^+} \frac{x}{x - 1}$;

   (b) $\lim_{x \to 1^-} \frac{x}{x - 1}$ (Hint: $\frac{-m}{1-m}$ can be expressed in the form $1 - \delta$);

   (c)* $\lim_{x \to \infty} \frac{\sqrt{x} - 5}{\sqrt{x} + 3}$;

   (d)* $\lim_{x \to \infty} \frac{\sqrt{x} - x}{\sqrt{x} + x}$;

   (e) $\lim_{x \to 0} \frac{\sqrt{x + 1}}{x}$.

   Check your results by definitions.